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Weighted Polynomial Approximation on the Real Line*

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We study polynomial approximation on the whole real line with weight $w = e^{-Q}$, where Q has polynomial growth at infinity. The following are the main problems considered: asymptotics for the Markov factors and for the rate of best approximation of $|x|$, Jackson-type estimates for the degree of best approximation of some classes of functions.

INTRODUCTION

Let Q be an even continuous function on the whole real line **R** positive on $(0, \infty)$, such that Q is increasing for sufficiently large x's and $Q(x)/\log x \to \infty$ as $x \to \infty$. In what follows any function Q is presumed to satisfy these properties, without explicitly mentioning it. (It should be noted that in most of the previous papers dealing with the subject, properties of Q are assumed to hold on the whole real line. We relax these and further conditions on Q by assuming most of them for sufficiently large x's only, thus emphasizing that in Q only the behavior at infinity is significant.)

In the last 10–15 years problems related to polynomial approximation on the whole real line with weight

$$w(x) := e^{-Q(x)}$$

have been widely investigated. The origin of these problems goes back to works of S. N. Bernstein. Let us introduce the rate of best weighted approximation

$$E_n^*(f, Q) := \inf_{p \in \Pi_n} \|w(x)(f(x) - p(x))\| \tag{1}$$

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of a function $f \in C(\mathbf{R})$ satisfying the condition $w(x)\,|f(x)| = o(1)$ $(|x| \to \infty)$. Here, as usual, Π_n denotes the set of algebraic polynomials of degree at most n, and $\|\cdot\|$ stands for the supremum norm on $\mathbf{R}$. Bernstein initiated the study of the quantity $E_n(|x|, Q)$. Let us mention here two basic results:

(i) $E_n(|x|, Q) \to 0$ as $n \to \infty$ if and only if

$$\int_1^\infty \frac{Q(t)}{t^2}\,dt = \infty$$

(see Akhiezer and Babenko [1]);

(ii) for $Q(x) = |x|$ we have $E_n(|x|, Q) = O((\log n)^{-1})$ (see Bernstein [3], p. 615).

Another central problem of weighted approximation concerns the so-called "Markov factors"

$$M_n(Q) := \sup_{p \in \Pi_n,\, p \not\equiv 0} \frac{\|w(x)\, p'(x)\|}{\|w(x)\, p(x)\|}. \tag{2}$$

The study of these factors was started by Freud [6] who showed that $M_n(Q) = O(n^{1-1/\alpha})$ if $Q(x) = |x|^\alpha$, $\alpha \geqslant 2$. Subsequently, Levin and Lubinsky [10] extended this result for $1 < \alpha < 2$, while Nevai and Totik [12] proved that[1] $M_n(Q) \sim \log n$ if $\alpha = 1$, and $M_n(Q)$ is bounded if $0 < \alpha < 1$. Moreover, Levin and Lubinsky [11] proved that if Q satisfies certain smoothness and growth conditions on $\mathbf{R}$ then $M_n(Q) = O(I_n(Q))$, where

$$I_n(Q) := \int_1^{Q^{\{-1\}}(n)} \frac{Q(t)}{t^2}\,dt \qquad (n \geqslant n_0). \tag{3}$$

(Here and in what follows $Q^{\{-1\}}$ stands for the inverse function of Q; by assumption, this inverse function exists for sufficiently large x's. Accordingly, some of the statements below hold only for sufficiently large n's; sometimes this will be indicated by writing $n \geqslant n_0$. It should be also noted that in [11] the authors use another equivalent form $I_n(Q) \sim \int_1^n dt/Q^{\{-1\}}(t)$.)

Some lower bounds for $M_n(Q)$ can be easily deduced with the help of the so-called Mhaskar–Rahmanov–Saff number $a_n = a_n(Q) > 0$ defined as the smallest interval $[-a_n, a_n]$ where the norm $\|w(x)\, p(x)\|$ is attained for any $p \in \Pi_n$, i.e. where this norm "lives." (It is easy to see that $a_n < \infty$. Under some additional conditions on Q, a_n can be given as a solution of

[1] In general, $\alpha_n \sim \beta_n$ will mean that the ratio α_n/β_n falls between two positive constants as $n \to \infty$.

a certain equation. We will not use this formula, since we do not want to restrict Q in this way.) Namely, with $\|x^n w(x)\| = x_0^n w(x_0)$ $(0 < x_0 \leqslant a_n)$ we get

$$\|nx^{n-1}w(x)\| \geqslant nx_0^{n-1}w(x_0) \geqslant \frac{n}{a_n}\, x_0^n w(x_0) = \frac{n}{a_n}\, \|x^n w(x)\|,$$

whence[2] by (2)

$$M_n(Q) \geqslant \frac{n}{a_n}. \tag{4}$$

For weights with polynomial growth (i.e., satisfying relations (13) below) $a_n \leqslant cQ^{[-1]}(n)$ (see our Lemma 3), which together with the last inequality leads to a lower bound $n/Q^{[-1]}(n)$ for $M_n(Q)$. (Here and in what follows $c, c_1, c_2, \ldots$ will always denote positive constants depending only on Q, not necessarily the same at each occurrence.) This lower bound is sharp when

$$\liminf_{x \to \infty} \frac{\log Q(x)}{\log x} > 1, \tag{5}$$

but is weaker than the upper bound $I_n(Q)$ when (5) fails to hold.

One of the main goals of this paper is to verify that under certain conditions on Q at infinity,

$$M_n(Q) \sim \frac{1}{E_n^*(|x|, Q)} \sim I_n(Q). \tag{6}$$

Here the lower bound for $M_n(Q)$ and the asymptotics for $E_n^*(|x|, Q)$ are new. Moreover, we give a new proof of the upper bound for $M_n(Q)$, which seems to be considerably simpler than the one given in [11], and also allows to relax substantially the restrictions imposed upon the weight Q in [11].

For functions $f \in C(\mathbf{R})$ satisfying $w(x)^\gamma |f(x)| = o(1)$ with some $0 < \gamma < 1$, we shall also investigate the order of magnitude of $E_n^*(f, Q)$, providing some Jackson-type estimates and discussing lower bounds as well.

The paper is divided into two sections. The first part is devoted to verifying (6), while the second is concerned with estimating the quantity $E_n^*(f, Q)$.

[2] We mention that an indication of a possible proof of (4) is mentioned in [11] (see the bottom of p. 1067).

1. Asymptotics for $M_n(Q)$ and $E_n^*(|x|, Q)$

We start with proving the following useful auxiliary statement.

Theorem 1. *There exists a constant $c > 0$ such that*

$$M_n(Q) \geq \frac{c}{E_n^*(|x|, Q)}. \tag{7}$$

Proof. Without loss of generality we may assume that $w(0) = 1$. Let $p_n(x) = \sum_{k=0}^{n} b_k x^k$ be such that

$$w(x) \mid |x| - p_n(x)| \leq E_n^*(|x|, Q) \qquad (x \in \mathbf{R}). \tag{8}$$

Then by simple iteration (see (2))

$$\begin{aligned} |b_k| &= \frac{|p_n^{(k)}(0)|}{k!} \leq \frac{M_n^k(Q)}{k!} \|w(x)\, p_n(x)\| \\ &\leq \left(\frac{eM_n(Q)}{k}\right)^k K \qquad (k = 0, 1, ..., n), \end{aligned} \tag{9}$$

where $K = 2\, \|w(x)\, |x|\, \|$. It is well known that there exists an absolute constant $\alpha > 0$ such that

$$\max_{|x| \leq 1} |\, |x| - q(x)| \geq \frac{\alpha}{n} \tag{10}$$

for every $q \in \Pi_n$ $(n = 1, 2, ...)$. Set $\alpha_1 := \max_{0 \leq x \leq 1} \{1/w(x)\}$ and

$$r_n := \left[\frac{\alpha}{2\alpha_1 E_n^*(|x|, Q)}\right] \leq \frac{n}{2}, \tag{11}$$

where [...] indicates integer part. We may assume that $r_n > 2eM_n(Q)$, since otherwise there is nothing to prove. (Note that $M_n(Q) \geq 2\, \|w\|/K$ for every $n \geq 1$.) Then using (8), (9), and (10) we have

$$\begin{aligned} \frac{\alpha}{r_n} &\leq \max_{|x| \leq 1} \left| |x| - \sum_{k=0}^{r_n - 1} b_k x^k \right| \leq \alpha_1 E_n^*(|x|, Q) + \sum_{k=r_n}^{n} |b_k| \\ &\leq \alpha_1 E_n^*(|x|, Q) + K \sum_{k=r_n}^{n} \left(\frac{eM_n(Q)}{k}\right)^k \\ &\leq \alpha_1 E_n^*(|x|, Q) + 2K \left(\frac{eM_n(Q)}{r_n}\right)^{r_n}. \end{aligned}$$

Combining this estimate with (11) we arrive at

$$\frac{\alpha}{2r_n} \leqslant 2K\left(\frac{eM_n(Q)}{r_n}\right)^{r_n},$$

i.e.,

$$M_n(Q) \geqslant \frac{r_n}{e}\left(\frac{\alpha}{4Kr_n}\right)^{1/r_n} \geqslant c_1 r_n.$$

In view of (11) this easily implies the needed statement.

Theorem 1 yields that in order to verify the asymptotic relations (6) it suffices now to provide upper estimates for $E_n^*(|x|, Q)$ and $M_n(Q)$.

We start with verifying the upper bound for $E_n^*(|x|, Q)$. We shall accomplish this with the help of the following basic result of Bernstein:

LEMMA 1 (Bernstein [3, p. 615]). *Let $R_{2n} \in \Pi_{2n}$, $R_{2n}(0) = 1$ be an even polynomial with positive coefficients and denote by ϱ_k $(k = 1, ..., 2n)$ the moduli of its roots. Then*

$$\inf_{p \in \Pi_n} \left\| \frac{|x| - p(x)}{\sqrt{R_{2n}(x)}} \right\| < 4\left(\sum_{k=1}^{2n} \frac{1}{\varrho_k}\right)^{-1}. \tag{12}$$

Furthermore we shall have to impose some mild restrictions on Q. Namely, it will be assumed that

$$\begin{cases} Q \in C^1[x_0, \infty) \text{ for some } x_0 > 0, \text{ and} \\ 0 < A(Q) := \liminf_{x \to \infty} \dfrac{xQ'(x)}{Q(x)} \leqslant B(Q) := \limsup_{x \to \infty} \dfrac{xQ'(x)}{Q(x)} < \infty. \end{cases} \tag{13}$$

It is easy to see from (13) that for sufficiently large x, $Q(x)/x^\alpha$ is increasing and $Q(x)/x^\beta$ is decreasing whenever $\alpha < A(Q) \leqslant B(Q) < \beta < \infty$. Moreover, using the relation

$$\log \frac{Q(b)}{Q(a)} = \int_a^b \frac{Q'(t)}{Q(t)}\, dt$$

we can also deduce from (13) that for sufficiently large x's and for any $\alpha < A(Q) \leqslant B(Q) < \beta < \infty$ and $c > 1$

$$c^\alpha Q(x) \leqslant Q(cx) \leqslant c^\beta Q(x), \tag{14}$$

i.e., $Q(cx) \sim Q(x)$ as $x \to \infty$. In addition, if Q satisfies (13) then a similar relation holds also with respect to its inverse $Q^{\{-1\}}$, and therefore we have, in particular, that for any $\alpha < B(Q)^{-1} \leqslant A(Q)^{-1} < \beta < \infty$ and $c > 1$

$$c^{\alpha} Q^{\{-1\}}(x) \leqslant Q^{\{-1\}}(cx) \leqslant c^{\beta} Q^{\{-1\}}(x) \tag{15}$$

for large x's. We shall frequently use the above properties of Q satisfying (13) without further reference.

THEOREM 2. *Let Q satisfy condition* (13). *Then*

$$E_n^*(|x|, Q) \leqslant \frac{c}{I_n(Q)} \qquad (n \geqslant n_0). \tag{16}$$

Proof. Let $\alpha < A(Q) \leqslant B(Q) < \beta < \infty$, and choose an integer $s \geqslant 1$ so that $2s > \beta$. Set

$$q_n(x) := \prod_{k=k_0}^{n} \left(1 + \frac{x^{2s}}{Q^{\{-1\}}(k)^{2s}}\right) \in \Pi_{2sn},$$

where k_0 is sufficiently large so that $Q^{\{-1\}}(k)$ is well defined for $k \geqslant k_0$. We shall verify that $|q_n(x)| = O(w(cx)^{-1})$ $(x \in \mathbf{R})$. Since $Q(x)/x^{\beta}$ is decreasing, evidently $Q^{\{-1\}}(x) > c_1 x^{1/\beta}$ for $x \geqslant x_0$. Hence

$$\sum_{k=k_0}^{\infty} Q^{\{-1\}}(k)^{-2s} \leqslant c_2 \sum_{k=k_0}^{\infty} k^{-2s/\beta} < \infty,$$

and therefore our claim is evident if $|x| \leqslant x_0$. Thus we may assume that $|x| \geqslant x_0$. Then

$$\begin{aligned} P_1 :=& \prod_{k_0 \leqslant k \leqslant Q(x)} \left(1 + \frac{x^{2s}}{Q^{\{-1\}}(k)^{2s}}\right) \\ \leqslant& \frac{x^{2sQ(x)}}{\prod_{k_0 \leqslant k \leqslant Q(x)} Q^{\{-1\}}(k)^{2s}} \prod_{k_0 \leqslant k \leqslant Q(x)} \left(1 + \frac{Q^{\{-1\}}(k)^{2s}}{x^{2s}}\right). \end{aligned}$$

Furthermore, using that $Q(x)/x^{\alpha}$ is increasing if $x \geqslant x_0$ we get integrating by parts

$$\begin{aligned} \prod_{k_0 \leqslant k \leqslant Q(x)} Q^{\{-1\}}(k) &= \exp \sum_{k_0 \leqslant k \leqslant Q(x)} \log Q^{\{-1\}}(k) \\ &\geqslant \exp\left(\int_{k_0}^{Q(x)} \log Q^{\{-1\}}(t)\, dt - \log x\right) \end{aligned}$$

$$= \exp\left(Q(x)\log x - \int_{Q^{\{-1\}}(k_0)}^{x} \frac{Q(u)}{u}\,du - c\log x\right)$$

$$\geqslant \exp\left(Q(x)\log x - \frac{Q(x)}{x^{\alpha}}\int_{x_0}^{x}\frac{du}{u^{1-\alpha}} - c_1\log x\right)$$

$$= \exp[Q(x)\log x + O(Q(x))].$$

Moreover,

$$\prod_{k_0\leqslant k\leqslant Q(x)}\left(1+\left(\frac{Q^{\{-1\}}(k)}{x}\right)^{2s}\right)$$

$$\leqslant \exp\left[x^{-2s}\int_{k_0}^{Q(x)} Q^{\{-1\}}(t)^{2s}\,dt + 1\right] \leqslant w(x)^{-1}.$$

Collecting the last there estimates yields

$$P_1 \leqslant x^{2sQ(x)}\exp[-2sQ(x)\log x + O(Q(x))] \leqslant w(x)^{-c}. \tag{17}$$

On the other hand, since $Q(x)/x^{\beta}$ is decreasing for $x \geqslant x_0$, and $2s > \beta$,

$$P_2 := \prod_{Q(x)<k\leqslant n}\left(1+\frac{x^{2s}}{Q^{\{-1\}}(k)^{2s}}\right) \leqslant \exp\sum_{Q(x)<k\leqslant n}\frac{x^{2s}}{Q^{\{-1\}}(k)^{2s}}$$

$$\leqslant \exp\left[x^{2s}\int_{Q(x)}^{\infty}\frac{dt}{Q^{\{-1\}}(t)^{2s}} + c\right] = \exp\left[x^{2s}\int_{x}^{\infty}\frac{dQ(u)}{u^{2s}} + c\right]$$

$$\leqslant \exp\left[x^{2s}\int_{x}^{\infty}\frac{2sQ(u)}{u^{2s+1}}\,du + c\right]$$

$$\leqslant \exp\left[2sx^{2s-\beta}Q(x)\int_{x}^{\infty}\frac{du}{u^{2s-\beta+1}} + c\right] = \exp(2sQ(x)+c). \tag{18}$$

Since $q_n = P_1P_2$ we obtain from (17), (18) and (14) that $|q_n(x)| \leqslant w(x)^{-c_1} \leqslant w(c_2x)^{-2}$ for $x \in \mathbf{R}$. This easily leads to a polynomial

$$\tilde{q}_n(x) := \prod_{k=k_0}^{n}\left(1+\frac{cx^{2s}}{Q^{\{-1\}}(k)^{2s}}\right) \in \Pi_{2sn}$$

such that

$$\sqrt{\tilde{q}_n(x)} \leqslant w(x)^{-1} \qquad (x \in \mathbf{R}). \tag{19}$$

Furthermore, if ϱ_k $(k=1, \ldots, 2s(n-k_0+1))$ are the moduli of the roots of $\tilde{q}_n$ then obviously

$$\frac{1}{2s}\sum_{k=1}^{2s(n-k_0+1)}\frac{1}{\varrho_k}=c\sum_{k=k_0}^{n}\frac{1}{Q^{\{-1\}}(k)}\geqslant c\int_{k_0}^{n}\frac{dt}{Q^{\{-1\}}(t)}$$

$$\geqslant c\int_{Q^{\{-1\}}(k_0)}^{Q^{\{-1\}}(n)}\frac{dQ(u)}{u}$$

$$=\frac{cn}{Q^{\{-1\}}(n)}+c\int_{Q^{\{-1\}}(k_0)}^{Q^{\{-1\}}(n)}\frac{Q(t)}{t^2}\,dt-c_1\geqslant c_2 I_n(Q). \tag{20}$$

Finally, using (12) (with $\tilde{q}_n$ instead of q_n), (19) and (20) we arrive at

$$E^*_{sn}(|x|, Q)\leqslant\frac{c}{I_n(Q)},$$

and this implies the statement of Theorem 2, since by (13) and (14) $I_n(Q)\sim I_{cn}(Q)$ for any fixed positive integer c.

Combining Theorems 1 and 2 we now have the needed upper bound for $E^*_n(|x|, Q)$ and lower bound for $M_n(Q)$. In order to complete the proof of the asymptotic relations (6) it is sufficient to show that $M_n(Q)\leqslant cI_n(Q)$. For this purpose we shall modify slightly condition (13). In fact, we shall preserve the lower bound

$$A(Q)=\liminf_{x\to\infty}\frac{xQ'(x)}{Q(x)}>0, \tag{21}$$

and impose a slightly stronger upper bound

$$C(Q):=\limsup_{x\to\infty}\frac{xQ''(x)}{Q'(x)}<\infty, \tag{22}$$

where it is assumed that $Q\in C^2[x_0, \infty)$ with some $x_0>0$. It is easy to see that $B(Q)\leqslant C(Q)+1$ for $B(Q)$ as in (13). Hence (21) and (22) imply (13). It should be also noted that condition (22) implies the polynomial growth of $Q'(x)$. Now we can formulate the following result.

THEOREM 3. *Let $Q\in C^2[x_0, \infty)$ for some $x_0>0$ and assume that* (21) *and* (22) *hold. Then*

$$M_n(Q)\leqslant cI_n(Q) \qquad (n\geqslant n_0). \tag{23}$$

Remark. As it was mentioned in the Introduction, the upper bound (23) for the Markov factor was given by Levin and Lubinsky [11] under stronger restrictions on Q. Namely, they assume that $Q \in C^2(0, \infty)$ and for every $x \in (0, \infty)$

$$-1 < A_1 \leqslant \frac{xQ''(x)}{Q'(x)} \leqslant A_2 < \infty. \tag{24}$$

It is easy to see that $A_1 + 1 \leqslant A(Q)$, whence the lower bound of (24) implies, in particular, that (21) holds. Thus restrictions (21) and (22) are weaker than (24) in two respects. First of all they remove any restrictions on Q for *small* x's, only the behavior of Q at infinity is relevant. Second, the lower bound (21) is weaker than the lower bound of (24). For example, if $Q(x) = x^3 + 10x \sin x + 20$ $(0 \leqslant x < \infty)$ then (21) and (22) hold but the lower bound in (24) fails even for large values of x. Finally, we mention that the proof of (23) below is based on a different approach and is significantly shorter than the one given in [11].

COROLLARY 1. *Under the conditions of Theorem* 3 *the asymptotic relations* (6) *hold.*

Our proof of Theorem 3 will be based on the following inequality which holds for any polynomial $p(z)$ (see [4], Theorem 6.5.4 on p. 92)

$$\log |p(z)| \leqslant \frac{|v|}{\pi} \int_{-\infty}^{\infty} \frac{\log |p(t)|}{(t-u)^2 + v^2}\, dt \qquad (z = u + iv \in \mathbf{C} \backslash \mathbf{R}). \tag{25}$$

We shall need an auxiliary result providing estimates for $|p_n'(x)|$ for $|x| \leqslant M$. Statements similar to Lemma 2 and Corollary 2 below can be found in Dzrbasyan [7], Theorem II.2.

LEMMA 2. *Let Q be arbitrary, $M > 0$ and $p_n \in \Pi_n$. Then we have*

$$|p_n'(x)| \leqslant c(M, Q) \left(I_n(Q) + \frac{n}{Q^{\{-1\}}(n)} \right) \|w(x) p_n(x)\|$$
$$(|x| \leqslant M,\ n \geqslant n_0), \tag{26}$$

where $c(M, Q) > 0$ is a constant depending only on M and Q.

Proof. Without loss of generality we may assume that $\|wp_n\| = 1$. We shall give an upper bound for $|p_n(z)|$ when $z = x + \varrho_n e^{i\varphi} (|\varphi| \leqslant \pi,\ |x| \leqslant M)$ with $\varrho_n := (I_n(Q) + n/Q^{\{-1\}}(n))^{-1} < c_0$ $(n \geqslant n_0)$, and then estimate $|p_n'(x)|$

using the Cauchy integral. Let $z=u+iv$, where $u=x+\varrho_n\cos\varphi$, $v=\varrho_n\sin\varphi$. Then by (25)

$$\begin{aligned}\log|p_n(z)| &\leqslant \frac{|v|}{\pi}\int_{-\infty}^{\infty}\frac{\log|p_n(t)|}{(t-u)^2+v^2}\,dt\\ &=\frac{|v|}{\pi}\left(\int_{|t|\leqslant 2|u|+1}+\int_{2|u|+1\leqslant|t|\leqslant Q^{\{-1\}}(n)}+\int_{|t|\geqslant Q^{\{-1\}}(n)}\right) \qquad (27)\\ &:=\frac{|v|}{\pi}(I_1+I_2+I_3).\end{aligned}$$

Since $|x|\leqslant M$ and $|u|\leqslant|x|+c_0\leqslant M+c_0$,

$$\frac{|v|}{\pi}I_1\leqslant\frac{|v|}{\pi}c_1(M,Q)\int_{-\infty}^{\infty}\frac{dt}{(t-u)^2+v^2}=c_1(M,Q). \qquad (28)$$

In I_2 we can use that $|t-u|>|t|/2$ leading to

$$\frac{|v|}{\pi}I_2\leqslant\frac{2|v|}{\pi}\int_1^{Q^{\{-1\}}(n)}\frac{Q(t)\,dt}{t^2/4+v^2}\leqslant c\varrho_n I_n(Q). \qquad (29)$$

Finally, using that

$$|q_n(x)|\leqslant\left(\frac{2|x|}{a}\right)^n\max_{|x|\leqslant a}|q_n(x)| \qquad (q_n\in\Pi_n,\ |x|\geqslant a)$$

(this follows from a rough estimate for the Chebyshev polynomials), we get

$$|p_n(t)|\leqslant\left(\frac{2et}{Q^{\{-1\}}(n)}\right)^n \qquad (|t|\geqslant Q^{\{-1\}}(n)),$$

whence for sufficiently large n's

$$\frac{|v|}{\pi}I_3\leqslant\frac{2|v|\,n}{\pi}\int_{Q^{\{-1\}}(n)}^{\infty}\frac{\log 2et/Q^{\{-1\}}(n)}{t^2}\,dt\leqslant\frac{cn\varrho_n}{Q^{\{-1\}}(n)}. \qquad (30)$$

Taking into account (27)–(30) yields that $|p_n(z)|\leqslant c(M,Q)$ for $z=x+\varrho_n e^{i\varphi}(|\varphi|\leqslant\pi)$. Now a standard application of the Cauchy formula for $p_n'(z)$ results in (26).

COROLLARY 2. *If Q is such that*

$$\int_1^{\infty}\frac{Q(t)}{t^2}\,dt<\infty, \qquad (31)$$

then for every $p_n \in \Pi_n$

$$|p_n'(x)| \leqslant c_1(M, Q) \, \|w(x) p_n(x)\| \qquad (|x| \leqslant M), \tag{32}$$

where $c_1(M, Q) > 0$ *depends only on* $M > 0$ *and* Q.

Proof. By the monotonicity of Q and (31), $I_n(Q) = O(1)$, and

$$\frac{n}{Q^{\{-1\}}(n)} = n \int_{Q^{\{-1\}}(n)}^{\infty} \frac{dt}{t^2} \leqslant \int_{Q^{\{-1\}}(n)}^{\infty} \frac{Q(t)\,dt}{t^2} = o(1),$$

i.e., (32) follows from (26).

Remark. The above corollary is somewhat more general than a result of Nevai and Totik [12] who show that $|p_n'(0)| \leqslant c(M, Q)$ if (31) holds. Furthermore, if in addition to (31) we also assume that

$$\sup_{x, y \in \mathbf{R}} [Q(x+y) - Q(x) - Q(y)] < \infty \tag{33}$$

then by a standard translation argument we can conclude that $\sup_{n \in \mathbf{N}} M_n(Q) < \infty$, i.e., the Markov factors are bounded. (In [12] the authors assume the concavity of Q but the weaker condition (33) suffices here.) Moreover, applying Theorem 1 we obtain the following criterion for the boundedness of the Markov factors.

COROLLARY 3. *Let* Q *satisfy* (33). *Then in order that* $\sup_{n \in \mathbf{N}} M_n(Q) < \infty$ *it is necessary and sufficient that* (31) *holds.*

For the proof of Theorem 3 we need an upper bound for the Mhaskar–Rahmanov–Saff number a_n mentioned in the Introduction.

LEMMA 3. *If* Q *satisfies* (13) *then* $a_n \leqslant c Q^{\{-1\}}(n)$ $(n \geqslant n_0)$.

It should be noted that the relation $a_n = O(Q^{\{-1\}}(n))$ also can be obtained from the known equation for the quantity a_n, but this requires additional assumptions on Q which are avoided in Lemma 3.

Proof. Assume that for a $p \in \Pi_n$, $\|wp\|$ is attained for $x_n \geqslant 0$, and set $u(x) = w(x) p(x)$, $x = Q^{\{-1\}}(n)\, y$, $x_n = Q^{\{-1\}}(n)\, y_n$, $\tilde{u}(y) = u(Q^{\{-1\}}(n)\, y)$, $\tilde{p}(y) = p(Q^{\{-1\}}(n)\, y)$. Then without loss of generality we may assume that $\max_{|y| \leqslant 1} |\tilde{p}(y)| = 1$. Hence

$$|\tilde{u}(y_n)| = \|\tilde{u}\| \geqslant \max_{|y| \leqslant 1} |\tilde{u}(y)| \geqslant e^{-n},$$

and by the Chebyshev inequality $|\tilde{p}(y)| \leqslant (2y)^n$ $(|y| \geqslant 1)$.

Assume that $y_n \geqslant 1$ (otherwise there is nothing to prove). By (14), for sufficiently large n's and any $0 < \alpha < A(Q)$, $Q(Q^{\{-1\}}(n)\, y_n) \geqslant n y_n^\alpha$. Then

$$e^{-n} \leqslant |\tilde{u}(y_n)| = w(Q^{\{-1\}}(n)\, y_n)\, |\tilde{p}(y_n)|$$
$$\leqslant w(Q^{\{-1\}}(n)\, y_n)(2y_n)^n \leqslant (2y_n e^{-y_n^\alpha})^n,$$

and thus $e^{y_n^\alpha} \leqslant 2ey_n$. Evidently, this implies that $0 \leqslant y_n \leqslant c_1$, and the lemma is proved.

Proof of Theorem 3. Assume that $|p_n(x)| \leqslant w(x)^{-1}$ $(x \in \mathbf{R})$. First we recall that (21) and (22) imply (13), hence, in particular, (14) and (15) also hold. This easily yields that

$$\frac{n}{Q^{\{-1\}}(n)} = n \int_{Q^{\{-1\}}(n)/2}^{Q^{\{-1\}}(n)} \frac{dt}{t^2} \leqslant c \int_{Q^{\{-1\}}(n)/2}^{Q^{\{-1\}}(n)} \frac{Q(t)}{t^2}\, dt \leqslant c I_n(Q), \tag{34}$$

i.e., in view of Lemma 2 we can restrict ourselves to sufficiently large x's. On the other hand, by the definition of the Mhaskar–Rahmanov–Saff number a_n, it suffices to give an upper bound for $|p_n'(x)|$ when $2 \leqslant x_0 \leqslant x \leqslant a_n$, with x_0 being sufficiently large. Let

$$\tilde{a}_n = 2cQ^{\{-1\}}(n), \qquad \varrho_n = \frac{I_{n_0}(Q)}{I_n(Q)}, \qquad z = x + \varrho_n e^{i\varphi} = u + iv \;\; (|\varphi| \leqslant \pi),$$

where c is the same as in Lemma 3. Then $0 < \varrho_n \leqslant 1$ $(n \geqslant n_0)$, and $1 \leqslant u \leqslant x + 1$. Using again (25) we have

$$\begin{aligned} \log |p_n(z)| &\leqslant \frac{|v|}{\pi} \int_{-\infty}^{\infty} \frac{\log |p_n(t)| - Q(u)}{(t-u)^2 + v^2}\, dt + Q(u) \\ &= \frac{|v|}{\pi} \left(\int_{-2\tilde{a}_n}^{u/2} + \int_{u/2}^{u} + \int_{u}^{3u/2} + \int_{3u/2}^{2\tilde{a}_n} + \int_{|t| \geqslant 2\tilde{a}_n} \right) + Q(u) \qquad (35) \\ &:= \sum_{i=1}^{5} I_i + Q(u). \end{aligned}$$

Applying Lemma 3 we can estimate I_5 similarly as in (30) which together with (34) yields $I_5 = O(1)$. Furthermore,

$$\begin{aligned} I_1 &\leqslant \frac{|v|}{\pi} \int_{-2\tilde{a}_n}^{u/2} \frac{Q(t) - Q(u)}{(t-u)^2 + v^2}\, dt \\ &= \frac{|v|}{\pi} \int_{-u/2}^{2\tilde{a}_n} \frac{Q(t) - Q(u)}{(t+u)^2 + v^2}\, dt \leqslant c\varrho_n \int_1^{c_1 Q^{\{-1\}}(n)} \frac{Q(t)}{t^2}\, dt = O(1). \end{aligned}$$

Similarly,

$$I_4 \leqslant c\varrho_n \int_{3u/2}^{2\bar{a}_n} \frac{Q(t)}{(t-u)^2}\, dt \leqslant c_1 \varrho_n \int_{3u/2}^{2\bar{a}_n} \frac{Q(t)}{t^2}\, dt = O(1).$$

Finally, with some ξ_t $(t/3 \leqslant 2u - t < \xi_t < t)$ we get

$$\begin{aligned} I_2 + I_3 &\leqslant \frac{|v|}{\pi} \left(\int_{u/2}^{u} + \int_{u}^{3u/2} \right) \frac{Q(t) - Q(u)}{(t-u)^2 + v^2}\, dt \\ &= \frac{|v|}{\pi} \int_{u}^{3u/2} \frac{Q(t) - 2Q(u) + Q(2u - t)}{(t-u)^2 + v^2}\, dt \\ &= \frac{|v|}{\pi} \int_{u}^{3u/2} Q''(\xi_t) \frac{(t-u)^2}{(t-u)^2 + v^2}\, dt. \end{aligned}$$

When x, and hence u, is large enough then by (22) (and (13))

$$Q''(\xi_t) \leqslant c_1 \frac{Q'(\xi_t)}{\xi_t} \leqslant c_2 \frac{Q(\xi_t)}{\xi_t^2} \leqslant c_3 \frac{Q(t)}{t^2}.$$

Using this in the previous estimate,

$$I_2 + I_3 \leqslant c\varrho_n \int_1^{c_1 Q^{[-1]}(n)} \frac{Q(t)}{t^2}\, dt = O(1).$$

Thus collecting the above estimates and substituting them into (35) implies for some η between x and u

$$\log |p_n(z)| \leqslant Q(u) + O(1) \leqslant Q(x) + \varrho_n Q'(\eta) + O(1).$$

Using again (13) and Lemma 3 we obtain just like in (34)

$$Q'(\eta) \leqslant c_1 \frac{Q(\eta)}{\eta} \leqslant c_2 \int_{\eta/2}^{\eta} \frac{Q(t)}{t^2}\, dt \leqslant c_3 \int_{1/2}^{a_n + 1} \frac{Q(t)}{t^2}\, dt \leqslant c_4 I_n(Q).$$

These inequalities imply

$$\log |p_n(z)| \leqslant Q(x) + O(1).$$

Hence by the Cauchy integral formula we obtain

$$w(x)\, |p_n'(x)| \leqslant \frac{c_0}{\varrho_n} = cI_n(Q).$$

This completes the proof of Theorem 3.

2. Order of Weighted Approximation on the Real Line

This section is devoted to Jackson-type estimates of the quantity $E_n^*(f, Q)$ defined by (1). In what follows, it will be more convenient to assume that $Q(x)$ is increasing on $(0, \infty)$ (and not just for sufficiently large x's). This additional restriction could be avoided but it would make the proofs more technical than desired. We shall study functions belonging to the class

$$C(\gamma, Q) := \{f \in C(\mathbf{R}) : f(x)\, w(x)^{\gamma} \to 0 \text{ as } |x| \to \infty\},$$

where γ is a fixed constant, $0 < \gamma < 1$. Moreover, we shall use the following modulus of continuity for an $f \in C(\gamma, Q)$

$$\omega_\gamma(f, h) := \sup_{\substack{x, y \in \mathbf{R} \\ |x-y| \leqslant h}} \frac{|f(x) - f(y)|}{w(x)^{-\gamma} + w(y)^{-\gamma}} \qquad (h > 0).$$

Evidently, for every $f \in C(\gamma, Q)$ we have $\omega_\gamma(f, h) \to 0$ as $h \to +0$. It is easy to see that

$$\omega_\gamma(f, h_1 + h_2) \leqslant 2\omega_\gamma(f, h_1) + 2\omega_\gamma(f, h_2),$$

and consequently

$$\omega_\gamma(f, \lambda h) \leqslant (2\lambda + 2)\, \omega_\gamma(f, h) \qquad (h, \lambda > 0). \tag{36}$$

Jackson-type estimates for this approximation problem in case of Freud-type weights Q (with some further restrictions on Q) are studied in detail in Ditzian and Totik [6]. A different modulus of continuity is introduced there which leads to a precise description of the rate of $E_n^*(f, Q)$ for Freud-type weights. In this section we shall give estimates for a very general class of weights Q. The only restriction we are imposing on Q is the rather mild growth condition

$$r := \liminf_{x, y \to \infty} \frac{Q(xy)}{Q(x) \log y} > 1. \tag{37}$$

However, we have to pay a price for this generality. Our method is based on the modulus of continuity introduced above which requires that $f \in C(\gamma, Q)$, and in addition, a superfluous logarithmic term appears in our estimate.

Let us now formulate the main result of this section.

THEOREM 4. *Let $0<\gamma<1$ and assume that Q satisfies* (13). *Then for every $f \in C(\gamma, Q)$ we have*

$$E_n^*(f, Q) \leqslant c_0 \omega_\gamma\left(f, \frac{Q^{[-1]}(n) \log n}{(1-\gamma) n}\right) + 3M_\gamma(f) e^{-c(1-\gamma)n} \tag{38}$$

where $M_\gamma(f) := \|w(x)^\gamma f(x)\|$.

Remark. When $Q(x)=|x|^\alpha$ $(\alpha>1)$, we have by (6)

$$E_n^*(|x|, Q) \sim n^{-(1-1/\alpha)}.$$

On the other hand (38) yields $E_n^*(|x|, Q)=O(n^{-(1-1/\alpha)} \log n)$, thus, in general, (38) is sharp apart from the $\log n$ factor. It should also be noted that (38) implies a convergence to 0 only if $Q^{[-1]}(n)=o(n/\log n)$.

In order to verify (38) we shall need some preliminaries.

Let $p_n^*(f, x) \in \Pi_n$ denote the polynomial of best approximation of f on $\mathbf{R}$, i.e.,

$$\|w(x)(f(x)-p_n^*(f, x))\| = E_n^*(f, Q).$$

Then there exist equioscillation points $x_0, \ldots, x_{n+1} \in \mathbf{R}$ such that

$$\begin{aligned} &w(x_i)(f(x_i)-p_n^*(f, x_i)) \\ &\quad = \varepsilon(-1)^i E_n^*(f, Q) \qquad (i=0, \ldots, n+1;\ \varepsilon=1 \text{ or } -1). \end{aligned}$$

By [13], p. 28, Theorem 2.5.1

$$E_n^*(f, Q) = \left| \frac{U\begin{pmatrix} f & 1 & \cdots & x^n \\ x_0 & x_1 & \cdots & x_{n+1} \end{pmatrix}}{U\begin{pmatrix} g & 1 & \cdots & x^n \\ x_0 & x_1 & \cdots & x_{n+1} \end{pmatrix}} \right|, \tag{39}$$

where

$$U\begin{pmatrix} \varphi_0 & \cdots & \varphi_{n+1} \\ x_0 & \cdots & x_{n+1} \end{pmatrix} := \det(\varphi_i(x_j))_{i,j=0}^{n+1},$$

and g is such that $w(x_i) g(x_i) = (-1)^i$ $(i=0, \ldots, n+1)$. Furthermore, denoting by B the $n \times (n+1)$ matrix with elements $x_{j+1}^i - x_j^i$ $(i=1, \ldots, n;\ j=0, \ldots, n)$ and by B_k the determinant of the matrix obtained from B by deleting its kth column we have, after some simple transformation

$$\begin{aligned} &U\begin{pmatrix} f & 1 & \cdots & x^n \\ x_0 & x_1 & \cdots & x_{n+1} \end{pmatrix} \\ &\quad = (-1)^n \sum_{k=0}^{n} (-1)^k \sum_{k=0}^{n} (-1)^k [f(x_{k+1}) - f(x_k)] B_k, \end{aligned}$$

and similarly

$$0 \neq U\begin{pmatrix} g & 1 & \cdots & x^n \\ x_0 & x_1 & \cdots & x_{n+1} \end{pmatrix}$$

$$= (-1)^{n+1} \sum_{k=0}^{n} [w(x_{k+1})^{-1} + w(x_k)^{-1}] B_k.$$

Thus we obtain from (39)

$$E_n^*(f, Q) = \left| \sum_{k=0}^{n} (-1)^k d_k [f(x_{k+1}) - f(x_k)] \right|, \tag{40}$$

where

$$d_k = \frac{B_k}{\sum_{j=0}^{n} [w(x_{j+1})^{-1} + w(x_j)^{-1}] B_j} \qquad (k = 0, ..., n). \tag{41}$$

(An expression similar to (40) is given in the periodic case in Babenko and Shalaev [2]; see also Bojanov [5], whose ideas are used in the proof of the next lemma.)

LEMMA 4. *With the previous notations, let*

$$\psi_n(x) := \begin{cases} (-1)^k d_k & \text{if } x \in [x_k, x_{k+1}] \ (k = 0, ..., n), \\ 0 & \text{otherwise.} \end{cases} \tag{42}$$

Then for every $p_{n-1} \in \Pi_{n-1}$ *we have*

$$\int_{\mathbf{R}} p_{n-1}(x)\, \psi_n(x)\, dx = 0. \tag{43}$$

Proof. For given $x_0, ..., x_{n+1}$, formula (40) gives an expression for the best approximation of f out of Π_n in the seminorm

$$\|f\|^* := \max_{0 \leqslant i \leqslant n+1} \{ w(x_i) |f(x_i)| \}.$$

Therefore for every $p_n \in \Pi_n$

$$0 = \sum_{k=0}^{n} (-1)^k d_k [p_n(x_{k+1}) - p_n(x_k)] = \int_{\mathbf{R}} \psi_n(x)\, p_n'(x)\, dx.$$

COROLLARY 4. *We have* $d_k > 0$ $(k = 0, ..., n)$ *and*

$$E_n^*(f, Q) \leqslant 4\omega_\gamma(f, \delta_n), \tag{44}$$

where

$$\delta_n := \sum_{k=0}^{n} d_k [w(x_k)^{-\gamma} + w(x_{k+1})^{-\gamma}](x_{k+1} - x_k). \tag{45}$$

Proof. By (41)

$$\sum_{k=0}^{n} d_k [w(x_k)^{-1} + w(x_{k+1})^{-1}] = 1. \tag{46}$$

The first claim follows now immediately from (42), (43) and (46). Furthermore, using (40), (36), (45) and (46) we have

$$\begin{aligned} E_n^*(f, Q) &\leqslant \sum_{k=0}^{n} d_k \, |f(x_{k+1}) - f(x_k)| \\ &\leqslant \sum_{k=0}^{n} d_k [w(x_k)^{-\gamma} + w(x_{k+1})^{-\gamma}] \, \omega_\gamma(f, x_{k+1} - x_k) \\ &\leqslant 2 \sum_{k=0}^{n} d_k [w(x_k)^{-\gamma} + w(x_{k+1})^{-\gamma}] \\ &\quad \times \left(\frac{x_{k+1} - x_k}{\delta_n} + 1 \right) \omega_\gamma(f, \delta_n) \leqslant 4\omega_\gamma(f, \delta_n). \end{aligned}$$

LEMMA 5. *Let* Q *satisfy* (37). *Then there exist constants* $c_1, c_2 > 0$ *such that for every* $p_n \in \Pi_n$ *and* n *large enough we have*

$$w(x) |p_n(x)| \leqslant e^{-c_2 n} \, \|w(x) p_n(x)\| \qquad (|x| \geqslant c_1 Q^{\{-1\}}(n)). \tag{47}$$

Proof. We may assume that $\|w(x) p_n(x)\| = 1$. Then for $|x| \leqslant Q^{\{-1\}}(n)$ we have $|p_n(x)| \leqslant e^n$. Using the Chebyshev inequality for the growth of polynomials we get

$$|p_n(x)| \leqslant \left(\frac{2e\,|x|}{Q^{\{-1\}}(n)} \right)^n \qquad (|x| \geqslant Q^{\{-1\}}(n)). \tag{48}$$

Assume that $c_1 \geqslant (2e)^{1/(r-1)}$ and n are sufficiently large so that by (37) we have for arbitrary $a \geqslant c_1$

$$Q(aQ^{\{-1\}}(n)) \geqslant rn \log a.$$

This and (48) yield that if $x = aQ^{[-1]}(n)$ (with arbitrary $a \geqslant c_1$), then for every $p_n \in \Pi_n$

$$w(x)|p_n(x)| \leqslant (2ea)^n a^{-rn} = (2ea^{1-r})^n.$$

Since $a \geqslant c_1 \geqslant (2e)^{1/(r-1)}$ $(r > 1)$ we obtain the statement of the lemma with a proper $c_2 > 0$.

The next statement concerning the "needle polynomials" is well known (see e.g., Kroó and Swetits [9], Lemma 3).

LEMMA 6. *Let* $-1 \leqslant a < b \leqslant 1$, $d := (b-a)/4$, $a_1 := a + d$, $b_1 := b - d$. *Then there exists a* $p_n \in \Pi_n$ *such that* $p_n \geqslant 0$ *on* $[-1, 1]$, $p_n \leqslant 1$ *on* $[-1, 1] \backslash [a, b]$ *and*

$$p_n(x) \geqslant e^{cn(b-a)} \qquad (x \in [a_1, b_1]). \tag{49}$$

Proof of Theorem 4. As above, we denote by $p_n^*(f)$ the best weighted approximant of $f \in C(\gamma, Q)$, and let x_j $(j = 0, ..., n+1)$ be the equioscillation points. We may assume that

$$r_n(f) := \max_{0 \leqslant j \leqslant n+1} |x_j| \leqslant c_1 Q^{[-1]}(n), \tag{50}$$

where c_1 is the same as in Lemma 5. Indeed, if $|x_j| \geqslant c_1 Q^{[-1]}(n)$ for some $0 \leqslant j \leqslant n+1$, then by Lemma 5

$$\begin{aligned} E_n^*(f, Q) &= w(x_j)\, |f(x_j) - p_n^*(f, x_j)| \\ &\leqslant M_\gamma(f)\, w(x_j)^{1-\gamma} + e^{-c_2 n} \|w(x)\, p_n^*(f, x)\| \\ &\leqslant M_\gamma(f) e^{-c_3(1-\gamma)n} + 2e^{-c_2 n} \|w(x)\, f(x)\| \leqslant 3M_\gamma(f)\, e^{-c(1-\gamma)n}. \end{aligned}$$

Hence (38) holds.

We now apply Lemma 6 with $[-1, 1]$ replaced by $[-r_n(f), r_n(f)]$ (this can be achieved by a homogeneous transformation of the variable). Then for any $[x_k, x_{k+1}]$ there exists a $p_{n,k} \in \Pi_{n-1}$ $(k = 0, ..., n)$ such that $p_{n,k} \geqslant 0$ on $[-r_n(f), r_n(f)]$, $p_{n,k} \leqslant 1$ on $[-r_n(f), r_n(f)] \backslash (x_k, x_{k+1})$, and by (50)

$$p_{n,k}(x) \geqslant \exp\left[\frac{cn(x_{k+1} - x_k)}{r_n(f)}\right] \geqslant \exp\left[\frac{c_2 n(x_{k+1} - x_k)}{Q^{[-1]}(n)}\right] \tag{51}$$

$$\left(x \in J_k := \left(\frac{3x_k + x_{k+1}}{4}, \frac{x_k + 3x_{k+1}}{4}\right),\ k = 0, ..., n\right).$$

Using these properties of $p_{n,k}$, the orthogonality relation (43), as well as (42) and (46), we get

$$\left|\int_{x_k}^{x_{k+1}} p_{n,k}(x)\,\psi_n(x)\,dx\right| = \left|\int_{\mathbf{R}\setminus[x_k,x_{k+1}]} p_{n,k}(x)\,\psi_n(x)\,dx\right|$$
$$\leqslant \int_{-r_n(f)}^{r_n(f)} |\psi_n(x)|\,dx$$
$$= \sum_{k=0}^{n} d_k(x_{k+1}-x_k)$$
$$\leqslant c \sum_{k=0}^{n} d_k[w(x_k)^{-1}+w(x_{k+1})^{-1}] = c.$$

This and (51) yield that

$$c \geqslant \left|\int_{x_k}^{x_{k+1}} p_{n,k}(x)\,\psi_n(x)\,dx\right| = d_k \int_{x_k}^{x_{k+1}} p_{n,k}(x)\,dx$$
$$\geqslant d_k \int_{J_k} p_{n,k}(x)\,dx \geqslant \frac{1}{2} d_k(x_{k+1}-x_k)\exp\left[\frac{c_2 n(x_{k+1}-x_k)}{Q^{(-1)}(n)}\right]. \qquad (52)$$

Now we shall give an upper bound for $E_n^*(f, Q)$ using Corollary 4. Set $X_k := \max(|x_k|, |x_{k+1}|)$ $(k=0, ..., n)$, $a := 2/(c_2(1-\gamma))$ (c_2 as in (52)), and let

$$K_1 := \left\{k : Q(X_k) \geqslant \frac{2}{1-\gamma} \log n,\ 0 \leqslant k \leqslant n\right\},$$
$$K_2 := \left\{k : k \notin K_1,\ x_{k+1}-x_k \leqslant a\,\frac{Q^{(-1)}(n)\log n}{n},\ 0 \leqslant k \leqslant n\right\},$$
$$K_3 := \{k : k \notin K_1 \cup K_2,\ 0 \leqslant k \leqslant n\}.$$

Obviously $\bigcup_{i=1}^{3} K_i = \{k : 0 \leqslant k \leqslant n\}$. Now by (45)

$$\delta_n = \sum_{i=1}^{3}\left(\sum_{k\in K_i} d_k[w(x_k)^{-\gamma}+w(x_{k+1})^{-\gamma}](x_{k+1}-x_k)\right) := \sum_{i=1}^{3} S_i. \qquad (53)$$

Furthermore, by (46) $d_k \leqslant w(X_k)$, whence

$$S_1 \leqslant 2\sum_{k\in K_1} w(X_k)^{1-\gamma}(x_{k+1}-x_k) \leqslant 2\sum_{k\in K_1}\frac{x_{k+1}-x_k}{n^2} \leqslant \frac{4r_n(f)}{n^2}.$$

Using (50) we have

$$S_1 \leqslant 4c_1 \frac{Q^{\{-1\}}(n)}{n^2}. \tag{54}$$

For the second sum we have by (46)

$$S_2 \leqslant \frac{aQ^{\{-1\}}(n)\log n}{n} \sum_{k \in K_2} d_k[w(x_k)^{-1} + w(x_{k+1})^{-1}]$$
$$\leqslant \frac{aQ^{\{-1\}}(n)\log n}{n}. \tag{55}$$

Finally, for the third sum we use (52)

$$S_3 \leqslant 4c \sum_{k \in K_3} \exp\left[\gamma Q(X_k) - c_2 \frac{n}{Q^{\{-1\}}(n)}(x_{k+1} - x_k)\right]$$
$$\leqslant 4c \sum_{k \in K_3} \exp\left[\frac{2\gamma}{1-\gamma}\log n - ac_2 \log n\right] \leqslant \frac{c_3}{n^2} \sum_{k \in K_3} 1 \leqslant \frac{c_3}{n}. \tag{56}$$

Thus we obtain from (53)–(56) that $\delta_n \leqslant cQ^{\{-1\}}(n)\log n/((1-\gamma)n)$. This together with (44) yields the desired result.

Theorem 4 provides a sufficiently sharp upper bound for $E_n^*(f, Q)$ when Q satisfies (13) with $A(Q) > 1$, because in this case (38) is exact up to the $\log n$ factor. On the other hand, it is well known that $E_n^*(f, Q) \to 0$ as $n \to \infty$ for every $f \in C(\gamma, Q)$ if and only if $I_n(Q) \to \infty$ as $n \to \infty$ (cf. Akhieser–Babenko [1]). Thus in order to have a complete result one would need to estimate $E_n^*(f, Q)$ in terms of $I_n(Q)$. Ideally, one would expect that (38) can be strengthened to

$$E_n^*(f, Q) \leqslant c\omega_\gamma\left(f, \frac{1}{I_n(Q)}\right). \tag{57}$$

Since $I_n(Q) \sim n/Q^{\{-1\}}(n)$ when (13) holds with $A(Q) > 1$, estimation (38) is rather close to (57) in this case.

The problem of verifying (57) appears to be quite hard. We can prove only the following weaker statement.

THEOREM 5. *Let Q satisfy (13) and assume that $0 < \gamma < 1$, $0 < \varepsilon < (1-\gamma)/(2-\gamma)$. Then for every $f \in C(\gamma, Q)$ there exists a $c := c(\varepsilon, f, Q) > 0$ such that*

$$E_n^*(f, Q) \leqslant c\omega_\gamma(f, I_n(Q)^{-(1-\gamma)/(2-\gamma)+\varepsilon}). \tag{58}$$

Proof. We shall approximate f by piecewise linear functions

$$l_N(x) := \sum_{k=-N}^{N} A_k\,|x-x_k| + Bx + C \qquad (x \in [-x_N, x_N]) \tag{59}$$

and use (16) for approximating $|x-x_k|$.

Set $N := [I_n(Q)^r]$, $p := Q^{(-1)}(\log N/(1-\gamma))$, where $r > 0$ will be chosen below. Let $x_k := kp/N$ ($k = -N, ..., N$) and let $l_N(x)$ be the piecewise linear function (59) which interpolates $f(x)$ at the x_k's. We also assume that $l_N(x)$ is extended from $[-p, p]$ to $\mathbf{R}$ as a constant (preserving continuity). It can be easily shown that the parameters A_k in (59) satisfy

$$\max_{|k| \leqslant N} |A_k| \leqslant \max_{|k| \leqslant N} \left| \frac{f(x_k) - f(x_{k+1})}{x_k - x_{k+1}} \right|. \tag{60}$$

For $x \in [x_k, x_{k+1}]$ we have

$$\begin{aligned} |f(x) - l_N(x)| &\leqslant \max_{x_k \leqslant y \leqslant x_{k+1}} f(y) - \min_{x_k \leqslant y \leqslant x_{k+1}} f(y) := f(\eta) - f(\xi) \\ &\leqslant \omega_\gamma(f, p/N)[w(x_k)^{-\gamma} + w(x_{k+1})^{-\gamma}] \leqslant c\omega_\gamma(f, p/N)\, w(x)^{-1} \\ &\qquad (\xi, \eta \in [x_k, x_{k+1}],\ k = -N, ..., N). \end{aligned}$$

When $|x| > p$ we have

$$|f(x) - l_N(x)| \leqslant |f(x)| + |l_N(x)| \leqslant cw(x)^{-\gamma} + |f(p)| + |f(-p)| \leqslant c_1 w(x)^{-\gamma}.$$

Combining the last two estimates we get, using that $w(p) = N^{1/(\gamma-1)}$,

$$\begin{aligned} w(x)\,|f(x) - l_N(x)| &\leqslant c[\omega_\gamma(f, p/N) + w(p)^{1-\gamma}] \\ &\leqslant c_1\,\omega_\gamma(f, p/N) \qquad (x \in \mathbf{R}). \end{aligned} \tag{61}$$

Furthermore, applying (16) with Q replaced by δQ ($\delta > 0$ sufficiently small) we obtain that for every $k = -N, ..., N$ there exists a $p_{k,n} \in \Pi_n$ such that

$$\|w(x-x_k)^\delta\,(|x-x_k| - p_{k,n}(x))\| \leqslant \frac{c}{I_n(\delta Q)} \leqslant \frac{c^*(\delta)}{I_n(Q)}, \tag{62}$$

with some $c^*(\delta) > 0$ depending on δ. Since Q satisfies (13) it follows that

$$Q(x-x_k) \leqslant Q(|x| + |x_k|) \leqslant Q(|x| + p) \leqslant c(Q(x) + Q(p)) \qquad (x \in \mathbf{R}). \tag{63}$$

Thus if δ is sufficiently small then we have from (62)

$$\|w(x)(|x-x_k| - p_{k,n}(x))\| \leqslant \frac{c^*(\delta)\,w(p)^{-c\delta}}{I_n(Q)} \leqslant \frac{c^*(\delta)\,N^{c_1\delta}}{I_n(Q)}.$$

Moreover by (60)

$$|A_k| \leqslant 2\omega_\gamma(f, p/N)\, w(p)^{-\gamma}\, N/p \leqslant c\omega_\gamma(f, 1/N)\, N^{1/(1-\gamma)}.$$

Thus there exists a $p_n^* \in \Pi_n$ for which with $r = (1-\gamma)/(2-\gamma) - \varepsilon_1$ $(0 < \varepsilon_1 < \varepsilon)$ and δ small enough

$$\|w(x)(l_N(x) - p_n^*(x))\| \leqslant c_\delta \frac{\omega_\gamma(f, 1/N)\, N^{(2-\gamma)/(1-\gamma) + c_1\delta}}{I_n(Q)} = O(\omega_\gamma(f, 1/N)).$$

Combining this with (61) and taking into account that

$$p = Q^{\{-1\}}\left(\frac{\log N}{1-\gamma}\right) = O((\log N)^\beta)$$

with some $\beta > 0$, we obtain (58).

Remark. For small ε and γ the estimation (58) approaches $O(I_n(Q)^{-1/2})$. For example, when $Q(x) = |x|$ then $I_n(Q) \sim \log n$, and (58) yields $O(\log^{-1/2+\varepsilon} n)$. Of course, when $Q(x) = |x|^\alpha$ with $\alpha > 1$ then $I_n(Q) \sim n^{1-1/\alpha}$ and (38) is much stronger than (58). Thus Theorem 5 provides a method for estimating $E_n^*(f, Q)$ which is useful for $Q(x) \sim |x|$. It also complements (38) in the sense that in case $I_n(Q) \to \infty$ $(n \to \infty)$ it yields $E_n^*(f, Q) \to 0$ $(n \to \infty)$.

As far as the authors know, Theorem 5 is the first Jackson-type estimate which covers also the "singular" case when $Q(x)$ is around $|x|$. (These weights are excluded in [6].) It has to be noted that the restriction $f \in C(\gamma, Q)$ is not very essential in Theorem 5; it is related to the specific modulus of continuity considered there. The method of Theorem 5 can be applied for estimating $E_n^*(f, Q)$ for arbitrary $f \in C(\mathbf{R})$ satisfying $\lim_{|x| \to \infty} w(x)|f(x)| = 0$, and Q satisfying (13), but this would require a more technical modulus of continuity. (Note that such a modification of Theorem 5 would include then the result of Akhiezer and Babenko [1].)

Let us mention now some possible lower bounds for $E_n^*(f, Q)$. One method is essentially shown in the proof of Theorem 1. We shall formulate now a somewhat more general statement. Let $\omega(f, h) := \sup_{x, y \in [-1, 1], |x-y| \leqslant h} |f(x) - f(y)|$ be the usual modulus of continuity of $f \in C[-1, 1]$, and $E_n(f) := \inf_{p \in \Pi_n} \max_{|x| \leqslant 1} |f(x) - p(x)|$ the error of best approximation on $[-1, 1]$.

THEOREM 6. *Assume that for an* $f \in C(\mathbf{R})$, $w(x)|f(x)| = O(1)$ *as* $|x| \to \infty$, *and for an infinite subset* $\Omega \subset \mathbf{N}$ *we have*

$$E_n(f) \geqslant c_1\, \omega\left(f, \frac{1}{n}\right) \qquad (n \in \Omega). \tag{64}$$

Then

$$E_n^*(f, Q) \geq c_2 \omega\left(f, \frac{1}{M_n(Q)}\right) \qquad (n \in \Omega). \tag{65}$$

The proof is very similar to that of Theorem 1 and therefore we omit the details.

Let us mention that when Q satisfies (13) with $A(Q) > 1$ then $M_n(Q) \sim n/Q^{\{-1\}}(n)$, i.e., (65) leads to

$$E_n^*(f, Q) \geq c_2 \omega\left(f, \frac{Q^{\{-1\}}(n)}{n}\right).$$

This lower bound should be compared with the upper estimate (38).

Another approach to estimating $E_n^*(f, Q)$ from below is similar to the Stechkin-type inequality on $[-1, 1]$

$$\omega(f, t) \leq ct \sum_{n=0}^{[1/\sqrt{t}]} nE_n(f). \tag{66}$$

To this end, assuming

$$\int_1^\infty \frac{Q(t)}{t^2}\, dt = \infty,$$

consider the Markov factor $M_n(Q)$, $n \in \mathbf{N}$ and a function $M(x)$ such that $M(n) \sim M_n(Q)$ and $M(x)$ is strictly increasing for large x's. Using this notation we can verify the following analogue of (66) for weighted approximation when $f \in C(1, Q)$ (i.e., $\gamma = 1$):

$$\omega_1(f, t) \leq ct \sum_{k=0}^{[M^{\{-1\}}(1/t)]} \frac{M(k)}{k+1} E_n^*(f, Q). \tag{67}$$

The proof of (67) is very similar to (66); one just has to use the Markov factor $M_n(Q)$ instead of the classical Markov constant n^2. For instance, for the Freud weights $Q(x) = |x|^\alpha$ $(\alpha > 1)$ we have $M(x) \sim x/Q^{\{-1\}}(x) = x^{1-1/\alpha}$ $(x > 0)$, implying

$$\omega_1(f, t) \leq ct \sum_{k=0}^{[t^{\alpha/(1-\alpha)}]} k^{-1/\alpha} E_k^*(f, Q).$$

Thus, in particular, if $E_n^*(f, Q) = O(k^{-\beta})$ with $0 < \beta < 1 - 1/\alpha$ then $\omega_1(f, t) = O(t^{\alpha\beta/(\alpha-1)})$. This again shows that (38) is exact apart from the log factor.

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